The Sequential Price Of Anarchy for Atomic Congestion Games*

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Abstract. In situations without central coordination, the price of anarchy relates the quality of any Nash equilibrium to the quality of a global optimum. Instead of assuming that all players choose their actions simultaneously, we consider games where players choose their actions sequentially. The sequential price of anarchy, recently introduced by Paes Leme, Syrgkanis, and Tardos [13], relates the quality of any subgame perfect equilibrium to the quality of a global optimum. The effect of sequential decision making on the quality of equilibria, depends on the specific game under consideration. We analyze the sequential price of anarchy for atomic congestion games with affine cost functions. We derive several lower and upper bounds, showing that sequential decisions mitigate the worst case outcomes known for the classical price of anarchy [2, 5]. Next to tight bounds on the sequential price of anarchy, a methodological contribution of our work is, among other things, a "factor revealing" linear programming approach we use to solve the case of three players.

1 Model and Notation

We consider atomic congestion games with affine cost functions. The input of an instance $I \in \mathcal{I}$ consists of a finite set of resources R, a finite set of players $N = \{1, \dots, n\}$, and for each player $i \in N$ a collection $\mathcal{A}_i$ of possible actions $A_i \subseteq R$. We say a resource $r \in R$ is chosen by player i if $r \in A_i$, where A_i is the action chosen by player i. By $A = (A_i)_{i \in N}$ we denote a possible outcome, that is, a complete profile of actions chosen by all players $i \in N$.

Each resource $r \in R$ has a constant activation cost $d_r \geq 0$ and a variable cost or weight $w_r \geq 0$ that expresses the fact that the resource gets more congested the more players choose it. The total cost of resource $r \in R$, for outcome A, is then $f_r(A) = d_r + w_r \cdot n_r(A)$, where $n_r(A)$ denotes the number of players choosing resource r in A. Given outcome A, the total cost of all resources chosen by player i is $\text{cost}_i(A) = \sum_{r \in A_i} f_r(A)$. Players aim to minimize their costs. The total cost over all players of an outcome A is denoted by $\text{cost}(A) = \sum_{i \in N} \text{cost}_i(A)$.

Note that this class of problems includes as a special case the celebrated network routing games as studied e.g. in [2, 15]. Another special case is singleton

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congestion games, where actions A_i are all singletons, $|A_i| = 1$. This model, and variants thereof, are also known as load balancing games and, with respect to the quality of equilibria, have a vast literature, e.g. [4, 11].

Pure Nash equilibria are outcomes $(A_i)_{i \in N}$ in which no player can decrease his costs by unilaterally deviating from choosing A_i. The price of anarchy *PoA* [9], measures the quality of any Nash equilibrium relative to the quality of a globally optimal allocation, *OPT*. Here *OPT* is an outcome minimizing the total costs over all players[1]. Our goal is to compare the quality of Nash equilibria to the quality of subgame perfect equilibria of an extensive form game as introduced in [10, 16]. We assume that the players choose their actions in an arbitrary, predefined order $1, 2, \ldots, n$, so that the i-th player must choose his action A_i, observing the actions of players preceding i, but not knowing the actions of the players succeeding him. A strategy S_i then specifies for player i the actions he chooses, one for each potential profile of actions chosen by his predecessors $1, \ldots, i-1$. We denote by S a strategy profile $(S_i)_{i \in N}$. The outcome $A(S) = (A(S)_i)_{i \in N}$ of a game is then the set of actions chosen by each player resulting from a given strategy profile S. We denote by $\text{cost}(S)$ the cost in the outcome $A(S)$.

Extensive form games can be represented in a game tree, with the nodes on one level representing the possible situations that a single player can encounter, and the edges emanating from any node representing the possible actions of that player in the given situation. The nodes of the game tree are also called information sets[2]. Subgame perfect equilibria are defined by Selten [16] as strategy profiles that induce Nash equilibria in any subgame of the game tree. The sequential price of anarchy of an instance I is defined by

$$\mathit{SPoA}(I) = \max_{S \in \mathit{SPE}(I)} \frac{\text{cost}(S)}{\text{cost}(\mathit{OPT}(I))}, \tag{1}$$

where $\mathit{SPE}(I)$ denotes the set of subgame perfect equilibria of instance I in extensive game form, and $\mathit{OPT}(\mathrm{I})$ denotes a social optimum outcome of I. The sequential price of anarchy of a class of instances $\mathcal{I}$ is defined as in [13] by $\mathit{SPoA}(\mathcal{I}) = \sup_{I \in \mathcal{I}} \mathit{SPoA}(I)$. Throughout the paper, when the class of instances is clear from the context, we write *PoA* and *SPoA*. Also, we use *OPT* and *SPE* to denote optimal and subgame perfect equilibrium outcomes respectively.

2 Related Work and Contribution

Recently, the sequential price of anarchy was introduced by Paes Leme et al. [13] as an alternative way to measure the costs of decentralization. Compared to the classical price of anarchy of Papadimitriou and Koutsoupias [9], it avoids

[1] Note that we consider a utilitarian global objective, that is, the global objective is to minimize the sum of the costs of all players. This is one of the standard models, yet different than the egalitarian makespan objective as studied, e.g., in [9].

[2] We deal with a game with perfect information, so all information sets are trivial, and subgame perfect equilibria can be computed by backward induction

the "curse of simultaneity" inherent in certain games [13]. More specifically, for machine cost sharing games, generic unrelated machine scheduling games and generic consensus games, the *SPoA* is smaller than the *PoA* [13]. However, for the latter two games, the 'generic' condition is indeed necessary [3]. Also, Bilò et al. [3] show that for many games myopic behaviour leads to better equilibria than the farsighted behaviour of subgame perfect equilibria. For throughput scheduling games, or more generally, set packing games, the *SPoA* is lower than the *PoA* [6]. For isolation games, however, the *PoA* is not worse than the *SPoA* in general [1]. These results leave a mixed impression, and lead to the natural question which classes of games possess an *SPoA* which is lower than the *PoA*. We address this question for atomic congestion games with affine cost functions. Congestion games were introduced by Rosenthal [14]. A special case is linear atomic congestion games, for which the price of anarchy is known to equal 2 in the case of two players, and 2.5 in the case of three or more players [2, 5].

Our contributions are both lower and upper bounds on the sequential price of anarchy for atomic congestion games with affine cost functions. For two and three players, we prove tight bounds of 1.5. and $2\frac{63}{488} \approx 2.13$, respectively. For $n = 4$ players, we derive a lower bound ≈ 2.46, yet we have not been able to derive a nontrivial constant upper bound (yet). In that respect note that, trivially, $SPoA \leq n$. We also consider the special case of singleton congestion games for which the *PoA* is 2.5 [4]. Here we give a parametric family of instances that yields a lower bound of $2 + 1/e \approx 2.37$, and we give an upper bound of $n - 1$. We substantially improve on these results for symmetric singleton congestion games, where we show that the *SPoA* equals 4/3, which matches the bound known for the *PoA* [8]. For each of the theorems in this paper we only give an outline of the proof. For full proofs and lower bound examples, we refer to our full paper[7].

3 General Linear Atomic Congestion Games

Theorem 1. $SPoA = 1.5$ *for atomic congestion games with two players and affine cost functions.*

We prove the theorem by considering only the relevant part of the game tree. For player 1, we only need to consider two actions; the action he chooses in a social optimum, and the action he chooses in a subgame perfect equilibrium. For player 2 we only need to consider 3 actions; the action he chooses in a social optimum, and his subgame perfect responses to both of player 1's actions. Therefore we only need to consider 6 outcomes. The general situation is shown in Figure 1. We lower bound the total cost in the subgame perfect equilibrium in terms of the total cost in the social optimum. The tight lower bound example uses only 2 actions per player and 3 resources in total.

Considering the simplicity of the lower bound example for 2 players, one might wonder whether it is possible to prove upper bounds in a more elegant fashion, for instance using smoothness or potential arguments. But, contrary to what one might expect, not every outcome of a subgame perfect equilibrium is

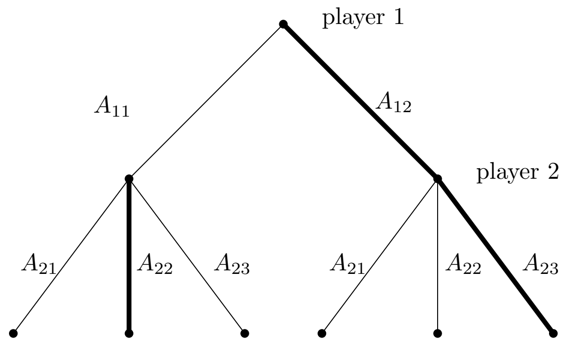

Fig. 1. All relevant actions in the game tree for 2 players. Fat lines correspond to subgame perfect actions.

a Nash equilibrium of the corresponding strategic form game.[3] In fact, we have constructed examples where it is subgame perfect for a player to choose an action that is strictly dominated in the corresponding strategic game. Therefore it is not easy to derive useful properties of *SPE* outcomes. Instead, for 3 players we use a linear programming (LP) approach.

Theorem 2. *SPoA* $= 2\frac{63}{488} \approx 2.13$ *for atomic congestion games with three players and affine cost functions.*

We first use simple combinatorial arguments to argue that a worst case instance is moderate in size. Specifically, we show that for any instance I, we can construct an instance I' with the same *SPoA* using only 2,3 and 7 actions for players 1,2 and 3 respectively. Moreover, I' has at most 4096 resources, one for every subset of all actions. Intuitively, the LP works as follows: It maximizes the *SPoA* over all instances with the properties described above. The only decision variables are the weights and constant costs of each of the 4096 resources. This completely determines the costs in all outcomes. We prespecify all subgame perfect actions in the game tree and normalize the costs in the social optimum to 1. Our only set of constraints enforces that in each node of the game tree, each subgame perfect action has a lower cost than any other action. Our objective is simply to maximize the total cost in the *SPE* outcome.

We have implemented this using the AIMMS modeling framework, and using CPLEX 12.5 we obtain an optimal solution with value $2\frac{63}{488}$. Given the techniques used so far, problems with $n > 3$ players become increasingly difficult. Extending the LP straightforwardly to the case with 4 players is problematic; using the same

[3] Note that both games have different strategy spaces: In the strategic form game both players have as strategy space their feasible actions, $\mathcal{A}_i$. In the extensive form game, however, the strategy space for the second player is more complex, as it specifies an action $A_2 \in \mathcal{A}_2$ for all information sets (= possible actions of player 1).

reasoning as in the thee player case, we would need to consider 43 actions for player 4, and 2^{55} resources. However, using ILP techniques, we have been able to construct lower bound examples for more than 3 players.

Theorem 3. *SPoA* ≥ 2.46 *for atomic congestion games with four players and affine cost functions.*

4 Singleton Linear Atomic Congestion Games

Next, we present results for the special case of singleton congestion games.

Theorem 4. *Asymptotically for* $n \to \infty$, *SPoA* $\geq 2 + \frac{1}{e} \approx 2.37$ *for singleton atomic congestion games with linear cost functions.*

The proof is by a parametric set of lower bound instances.

Theorem 5. *For singleton atomic congestion games with affine cost functions, SPoA* $\leq n - 1$.

The proof is by contradiction. Suppose the theorem does not hold, then for some instance I, *SPoA*$(I) > n - 1$. Therefore there exists at least one player i for whom $\text{cost}_i(SPE) \geq (n-1)\,\text{cost}_i(OPT)$. With this, we can construct a contradiction. However, note that this bound is close to the trivial upper bound n that holds for general congestion games.

Theorem 6. *For symmetric singleton atomic congestion games with affine cost functions, SPoA* $= 4/3$.

To prove the theorem, we first prove that any *SPE* outcome of a sequential game is also an *NE* outcome of the corresponding strategic game. Note that this is not trivial or even true in a more general setting as mentioned in Section 2. Also note that the theorem is not implied by results in [12]; for the non-generic case, Milchtaich proves only the existence of an *SPE* outcome that is an *NE* outcome. Intuitively our proof is as follows: we show that for any player i for whom there exists a resource r' in an *SPE* outcome that is less costly than the resource r he chose, we can find a successor j for whom there exists a less costly resource in the *SPE* outcome in the subgame where player i chooses r'. With this, we construct a contradiction. The theorem follows from the fact that *PoA* $= 4/3$, as shown in [8], and a matching lower bound example. Figure 2 gives an overview.

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congestion game	# players	*PoA*	*SPoA*
general	$n = 2$	$2[5, 2]$	1.5
general	$n = 3$	$2.5[5, 2]$	$2\frac{63}{488}$
general	$n = 4$	$2.5[5, 2]$	> 2.46
singleton	$n \geq 3$	2.5[4]	$\leq n - 1$
singleton	$n \to \infty$	2.5[4]	$\geq 2\frac{1}{e}$
singleton & symmetric	$n \geq 2$	4/3 [8]	4/3

Fig. 2. Results for the *SPoA* in comparison to the *PoA*.

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