Algorithms for Optimal Price Regulations

Alexander Grigoriev[1], Joyce van Loon[1⋆], and Marc Uetz[2]

[1] Maastricht University, Quantitative Economics,
P.O.Box 616, NL–6200 MD Maastricht, The Netherlands
{a.grigoriev,j.vanloon}@ke.unimaas.nl

[2] University of Twente, Applied Mathematics,
P.O. Box 217, NL–7500 AE Enschede, The Netherlands
m.uetz@utwente.nl

Abstract. Since summer 2007, mobile phone users in the European Union (EU) are protected by a ceiling on the roaming tariff when calling or receiving a call abroad. We analyze the effects of this price regulative policy, and compare it to alternative implementations of price regulations. The problem is a three-level mathematical program: The EU determines the price regulative policy that maximizes overall social welfare, the telephone operator sets profit-maximizing prices, and customers choose to accept or decline the operator's offer. The first part of this paper contains a polynomial time algorithm to solve such a three-level program. The crucial idea is to partition the polyhedron of feasible price regulative parameters into a polynomial number of smaller polyhedra such that a certain primitive decision problem can be written as an LP on each of those. Then the problem can be solved by a combination of enumeration and linear programming. In the second part, we analyze more specifically an instance of this problem, namely the price regulation problem that the EU encounters. Using customer-data from a large telephone operator, we compare different price regulative policies with respect to their social welfare. On the basis of the specific social welfare function that we use, we observe that other price regulative policies or different ceilings can improve the total social welfare.

Keywords: Pricing problems, three-level program optimization, social welfare maximization, EU roaming regulation.

1 Introduction

It is of major importance to the European Union (EU) that European companies, governments and citizens play an important role at the realization of a world-economy based on knowledge. The EU tries to stimulate the development and use of new information and communication technology, and to enlarge the level of competition of the EU compared to other markets, e.g. United States and Japan.

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An important element of the European policy is to assure that ICT-services are available and affordable for everyone. This contains for example telephony, fax, internet and free emergency numbers[3]. However, especially the prices for making and receiving calls abroad, referred to as *roaming*, have been extremely high in the EU recently. A warning did not lead to a decrease in prices, and therefore the European Commission uses *price regulation* to force lower prices and more transparency in the market [1]. Currently, the EU considers the same instrument for *data roaming*, since the situation mirrors the one for voice roaming back in 2007. We analyze both algorithmically and practically the effects of the current price regulation on the *social welfare*, and compare it to alternative regulations.

We regard a general model in which a government tries to maximize *social welfare* through price regulation. This regulation should bound the producer (not necessarily a telephone operator) in setting the prices so as to protect the customers in the market. The producer determines the price of *items*. An item is not necessarily a physical product, but can for example also be a minute calling, internet connection, shipping, etc. We use a model in which there is only one producer that determines the pricing strategy, under given and known market competition. Obviously, there exist markets in which multiple producers operate and need to share a set of customers. However, for this research we make the simplifying assumption that, under given market regulations, producers reach an optimal market price, so we identify them with just one single producer.

1.1 Model

Let $K = \{1, \ldots, m\}$ be the set of distinct item types a given producer wants to price. Let p_k be the price of items of type $k \in K$ that needs to be determined by the producer. Let $J = \{1, \ldots, n\}$ be a set of potential customers. Every customer $j \in J$ has demand d_{jk} for item $k \in K$, which is the number of times customer j wants to purchase item k. For example, if item k represents calling abroad for one minute, then d_{jk} is the number of minutes j wants to call abroad. Or, if k is the start up of a process, then d_{jk} is 1 if j prefers to start the process and 0 otherwise. Every customer j requests a *contract* from the producer, which is specified by the total demand vector $(d_{j1}, \ldots, d_{jm})$. Once the item prices are determined, the price of the contract, $p(j)$, is defined by the following *affine function*

$$p(j) = d_{j0} + d_{j1}p_1 + \cdots + d_{jm}p_m\,, \qquad j \in J\,. \tag{1}$$

Note that the price of a contract is *personal*, due to a potential ‘entrance fee’ d_{j0}, and because it depends on the demand $(d_{j1}, \ldots, d_{jm})$ of a single customer. The pricing regime as defined in (1) is referred to as *affine pricing* in the economic literature; it was also discussed by Grigoriev et al. in [11]. It is probably a more realistic model than the single item pricing model that was discussed in many papers on algorithmic profit maximizing pricing problems [2, 3, 7–10, 12–14]. A solution to the problem is a price $p(j)$ for every customer $j \in J$, which

[3] From: `http://www.europa-nu.nl`.

is determined through a vector of item prices $p = (p_1, \ldots, p_m)$ as given in (1). Every customer decides whether to accept this contract or not. Hereto, she sets binary variable w_j to 1 if she accepts, and 0 otherwise, in order to maximize her personal objective, denoted by $f_j^{\mathrm{C}}(p)$. In this paper, we assume a linear objective function. For example, think of $f_j^{\mathrm{C}}(p) = b_j - p(j)$, where b_j represents the valuation of customer j. Let $w = (w_1, \ldots, w_n)$ denote the strategies of all customers. Customers that accept the contract are referred to as *winners*, and the set of winners is defined by $W = \{j \in J : w_j = 1\}$. We assume that all items are available in unlimited supply, which is true for digital items for example. Thus any solution will be trivially envy-free. We assume, e.g. through market research, that we know the customers' demands d_{jk} for all items $k \in K$, and the valuations b_j, for all customers $j \in J$. Therefore, we are faced with a purely algorithmic problem in contrast to mechanism design problems where the valuations are private information to the customers.

The government protects the customers by means of regulative constraints. Let R denote the set of constraints imposed by the government. Throughout this paper, we assume that the number of regulations $|R|$ is constant. Every constraint $r \in R$ is defined by $g_r(p, \alpha_r) \leq 0$, where $\alpha = (\alpha_1, \ldots, \alpha_{|R|})$ is a vector of price regulative parameters determined by the government. For example, a ceiling on item price p_k is implemented by letting $g_k(p, \alpha_k) = p_k - \alpha_k$. In this paper, function $g_r(p, \alpha_r)$ is restricted to be a *linear* function in α. We introduce a bilevel program [15] in which the producer maximizes his objective $f^{\mathrm{P}}(p, w)$ (e.g. revenue minus production costs) such that price vector $p = (p_1, \ldots, p_m)$ satisfies the price regulative constraints. Every customer maximizes her objective $f_j^{\mathrm{C}}(p)$.

$$
\begin{array}{lccl}
2\textsc{lp}: & \max_{p \in \Re_+^m} & f^{\mathrm{P}}(p, w) & \\
& \max_{w_j \in \{0,1\}} & f_j^{\mathrm{C}}(p) w_j & \forall j \in J \\
& s.t. & g_r(p, \alpha_r) \leq 0 \; \forall r \in R &
\end{array}
$$

In the above mathematical program, price regulative parameters α and constraints $g_r(p, \alpha_r)$ are assumed to be given. This bilevel problem can be solved in polynomial time for a constant number of distinct items m, by a simple enumerative algorithm [11]. In this paper, we propose a *three-level* program where, on top of the two levels given by producer and customers, the government maximizes social welfare by modifying the price regulative parameters α. Also, instead of simply forbidding the violation of price regulative constraints, we introduce taxes $\tau = (\tau_1, \ldots, \tau_{|R|})$. That is, if the producer's prices violate regulation $g_r(p, \alpha_r) \leq 0$, then he pays a penalty over the additional profit he receives by this violation. The actual penalty for violating regulation r is denoted by $f_r^{\mathrm{TAX}}(p, w, \alpha, \tau) = g_r(p, \alpha_r)^+ \cdot \tau_r \cdot \bar{g}_r(w)$, where $g_r(p, \alpha_r)^+ = \max\{0, g_r(p, \alpha_r)\}$ is the amount of violation, τ_r is the tax (or penalty), and finally function $\bar{g}_r(w)$ needs to be specified for each type of price regulative constraint r. For example, if the regulation is a ceiling $g_k(p, \alpha_k) = p_k - \alpha_k$, it sounds reasonable to ask tax τ_k for each euro earned by violating the ceiling. Then $\bar{g}_r(w)$ would have to be defined as the total demand of all winners, $\sum_{j \in J} d_{jk} w_j$. Let us denote the total tax payment for a producer by

$$f^{\text{TAX}}(p,w,\alpha,\tau) = \sum_{r\in R} f_r^{\text{TAX}}(p,w,\alpha,\tau) = \sum_{r\in R} g_r(p,\alpha_r)^+ \cdot \tau_r \cdot \bar{g}_r(w)\,. \tag{2}$$

The mathematical program (3LP) below shows the general structure of the three-level program that we consider.

$$\begin{array}{lcr} 3\text{LP} : \max_{\alpha,\tau\in\Re_+^{|R|}} & f^{\text{G}}(p,w,\alpha,\tau) & \\ \max_{p\in\Re_+^m} & f^{\text{P}}(p,w) - f^{\text{TAX}}(p,w,\alpha,\tau) & \\ \max_{w_j\in\{0,1\}} & f_j^{\text{C}}(p)w_j & \forall j \in J \end{array}$$

If the government's objective $f^{\text{G}}(p,w,\alpha,\tau)$ is *monotone* in the prices then we show that we can solve this program in polynomial time, given that the number of items m and the number of regulative constraints $|R|$ are constant. Monotonicity in prices is a reasonable assumption in any realistic economic setting.

Note that the lower two levels of 3LP can be seen as the Lagrangian of the bilevel program 2LP, and a strict price regulation as in the bilevel model (that is, forbidding to violate the regulative constraints) can still be implemented by letting τ_r be arbitrarily large for all $r \in R$ (then τ is not strictly a tax, but just an arbitrary penalty).

It is known that problem 2LP with a non-constant number of items m is hard to approximate within a semi-logarithmic factor in the number of customers n [8]. This means, that solving the three-level program 3LP has the same complexity already if the government's objective is equal to the producer's objective. As in [11], in this paper we make the assumption that the number of distinct item types m is constant. We also assume that the number of regulations $|R|$ is constant. These are reasonable assumptions particularly for the applications that we aim at, since there the number of item types is very small (for example, price per minute for a call received, a call placed, and price per SMS). For a small number of items, the number of regulations is also assumed to be small.

1.2 Our results

For a constant number of items m and regulative constraints $|R|$, we present a polynomial time algorithm to solve three-level program 3LP, under the restrictions that the government's objective function is monotone in the item prices, and the price regulative constraints $g_r(p,\alpha_r)$ are linear in α for all regulations $r \in R$.

We explicitly define all functions in three-level program 3LP to optimize the social welfare for the specific problem faced by the European Union in regulating the roaming charges. We present polynomial time algorithms to find the optimal social welfare for the current EU policy (Theorem 3), optimization of the tax level only (Theorem 4), and optimization of the regulative parameters only (Theorem 5). After the theoretical results and description of the algorithms we perform an extensive practical study in Section 4 to verify practical feasibility of the approach and to evaluate the result of different scenarios for implementing price regulations. Here, we use the actual price regulations set by the European Commission and investigate the EU policy in terms of social welfare.

2 Parameter and Tax Level Optimization

Consider three-level program 3LP, as specified in the introduction. In this section, we propose an algorithm to solve this program for which the government's objective function is monotone in the prices, and the regulations $g_r(p, \alpha_r)$, $r \in R$, are linear in α (note that the function does not have to be linear in p).

Definition 1. *The set V of vertices is defined as all price vectors $p = (p_1, \ldots, p_m)$ defined by m linearly independent constraints out of the $n+m$ constraints $f_j^{\mathrm{C}}(p) = 0$, $j \in J$, and $p_k = 0$, $k \in K$.*

The government's objective function f^{G} is monotone in the prices. Therefore, we straightforwardly derive the following theorem.

Theorem 1. *For any given vectors w, α and τ, the optimal price vector $p = (p_1, \ldots, p_m)$ for the producer can only be at a vertex as defined in Definition 1.*

We propose an algorithm that solves problem 3LP by optimizing price regulative parameters α and the taxes τ simultaneously. Thereto, we partition the polyhedron of the price regulative parameters α into a polynomial number of smaller polyhedra, and solve a linear programming problem in each of those. These linear programs are defined in such a way that we can verify if a given price vector constitutes the producer's optimum prices. This decision problem is in general non-linear. The trick here is to define the partition such that this decision problem becomes linear inside each of the small polyhedra. The optimum solution of a three-level mathematical program is eventually obtained by enumeration over all polyhedra and vertices, and evaluating the social welfare in each of them. Consider the following decision problem.

Problem 1 *Are there price regulative vectors α and τ, such that vertex $v \in V$ with price vector $p^{(v)}$ maximizes the objective function $f^{\mathrm{P}} - f^{\mathrm{TAX}}$ for the producer?*

The main idea for the solution to Problem 1: Consider some arbitrary vertex $v \in V$, we would like to write constraints expressing the fact that vertex v maximizes the objective, namely $f^{\mathrm{P}}(p^{(v)}, w^{(v)}) - f^{\mathrm{TAX}}(p^{(v)}, w^{(v)}, \alpha, \tau) \geq f^{\mathrm{P}}(p^{(u)}, w^{(u)}) - f^{\mathrm{TAX}}(p^{(u)}, w^{(u)}, \alpha, \tau)$ for all vertices $u \in V$. By definition, $f^{\mathrm{TAX}}(p^{(v)}, w^{(v)}, \alpha, \tau)$ is nonlinear in the price regulative parameters. To linearize f^{TAX} we introduce a subdivision of $\Re_+^{|R|}$ into polyhedra A_l, $l = 1, \ldots, L$. For a given vertex $p^{(v)}$, $v \in V$ and by linearity of $g_r(p^{(v)}, \alpha_r)$, there is a unique value for α_r, say a_r^v, where the sign of $g_r(p^{(v)}, \alpha_r)$ changes from ≤ 0 to > 0. Doing this for all vertices $v \in V$ and all regulative constraints $r \in R$, we define a rectangular subdivision in $\Re_+^{|R|}$ for possible α's by $\alpha_r = a_r^v$. On each such defined α-rectangle A_l, we may now compare the producer's objective in vertex $v \in V$

to the objective in all other vertices $u \in V \setminus v$. To do this, we split the penalty function f_r^{TAX}, $r \in R$, into two parts for every vertex $v \in V$ and polyhedron A_l: $x_r^{(v,l)}$ incorporates all terms in the regulative constraint not multiplied with α_r and $y_r^{(v,l)}$ incorporates all terms multiplied with α_r. Therefore,

$$f_r^{\text{TAX}}(p^{(v)}, w^{(v)}, \alpha, \tau) = g_r(p, \alpha_r)^+ \cdot \tau_r \cdot \bar{g}_r(w) = x_r^{(v,l)}\tau_r + y_r^{(v,l)}\alpha_r\tau_r, \quad \forall\, r \in R.$$

We can derive a solution to Problem 1 on A_l by solving the following mathematical program for every vertex $v \in V$.

$$\begin{aligned} f^{\text{P}}(p^{(v)}, w^{(v)}) - \left(\sum_{r\in R} x_r^{(v,l)}\tau_r + y_r^{(v,l)}\alpha_r\tau_r\right) &\geq \\ f^{\text{P}}(p^{(u)}, w^{(u)}) - \left(\sum_{r\in R} x_r^{(u,l)}\tau_r + y_r^{(u,l)}\alpha_r\tau_r\right) & \quad \forall\, u \in V \\ \check{\alpha}_r^{(l)} \leq \alpha_r \leq \hat{\alpha}_r^{(l)} & \quad \forall\, r \in R \\ \tau_r \geq 0 & \quad \forall r \in R. \end{aligned}$$

This quadratic program can be linearized by simple variable substitution $\phi_r = \alpha_r\tau_r$ for all $r \in R$. Therefore, for every vertex $v \in V$ and polyhedron A_l, Problem 1 becomes a linear program (LP1) with variables τ_r and ϕ_r for all $r \in R$.

$$\begin{aligned} \text{LP1}: \sum_{r\in R} \left(x_r^{(v,l)} - x_r^{(u,l)}\right)\tau_r + \left(y_r^{(v,l)} - y_r^{(u,l)}\right)\phi_r &\leq \\ f^{\text{P}}(p^{(v)}, w^{(v)}) - f^{\text{P}}(p^{(u)}, w^{(u)}) & \quad \forall\, u \in V \\ \check{\alpha}_r^{(l)}\tau_r \leq \phi_r \leq \hat{\alpha}_r^{(l)}\tau_r & \quad \forall\, r \in R \\ \tau_r \geq 0 & \quad \forall r \in R. \end{aligned}$$

On any polyhedron A_l this linear program is either infeasible, suggesting that there are no price regulative parameters in A_l that makes v the solution that maximizes the producer's objective, or otherwise we obtain corresponding price regulative parameters in A_l. Eventually, a straightforward algorithm enumerating all vertices $v \in V$, checking feasible solutions for α's in $A_l, l = 1, \ldots, L$, and picking the one that achieves the maximal social welfare, provides an optimal solution to the three-level program. Since the number of items m is constant, we have a polynomial number of vertices. As the number of regulative constraints $|R|$ is constant, we have only a polynomial number of polyhedra in $\Re_+^{|R|}$. For every polyhedron and vertex, we solve linear program LP1, deriving the following theorem.

Theorem 2. *Three-level program* 3LP *admits a polynomial time algorithm if the number of items m and the number of regulative constraints $|R|$ are constant.*

In this section, we use price regulative constraints in which there is one particular α_r and τ_r for every constraint $r \in R$. However, the above described algorithm can easily be adapted for other cases, that is, there might be multiple α parameters or tax levels τ in one constraint, or multiple constraints can be subjected to the same tax level τ or contain the same α.

3 Optimization of European Regulation

In this section, we explicitly define functions f^{G}, f^{P}, f^{C} and f^{TAX} to solve the problem faced by the European Union regarding the regulation on roaming. First of all, the regulations set ceilings on the prices. That is, $R = K$, and $g_k(p, \alpha_k) = p_k - \alpha_k$ for all items $k \in K$. The objective of the producer is to maximize the profit, defined as revenue minus costs. The revenue is the total payment by all winning customers. As for the costs, let c_k be the cost of providing one unit of k to a customer. The producer's cost to serve customer j is denoted by $c(j) = d_{j1}c_1 + \cdots + d_{jm}c_m$. The customers accept a contract if its price does not exceed the valuation, that is, the objective of customer $j \in J$ is defined as $(b_j - p(j))w_j$. If $p(j) \le b_j$ this function is maximized by setting $w_j = 1$ and thus accepting the contract, and 0 otherwise.

In the simplest setting, the government regulates the prices by forbidding to violate the constraint $p_k \le \alpha_k$ for every item $k \in K$. We model this by the following bilevel program.

$$\begin{array}{lll} \max_{p \in \Re^m_+} & \sum_{j \in J}(p(j) - c(j))w_j & \\ \max_{w_j \in \{0,1\}} & (b_j - p(j))w_j & \\ s.t. & p_k \le \alpha_k & \forall\, k \in K \end{array}$$

To find the prices that will lead to the optimal profit for the producer, we use the affine pricing algorithm introduced in [11], in which we incorporate the price regulative constraints.

Theorem 3 ([11]). *For given price regulative constraints $p_k \le \alpha_k$ which must not be violated, profit maximizing prices can be computed in polynomial time, given that the number of distinct item types m is constant.*

As already discussed in the introduction, we study if there are other price regulative strategies that might lead to an increase in *social welfare*. Thus, let us first proceed with a definition of the social welfare function we believe to be appropriate for modeling the roaming regulation problem. According to utilitarians such as Jeremy Bentham and John Stuart Mill, society should aim to maximize the total *utility* of individuals, aiming for "the greatest happiness for the greatest number". Thus, the government strives to set the price regulative parameters so as to maximize the social welfare, defined as the sum of utilities. The utility of the producer is the total revenue minus the costs. We assume that the producer is risk-neutral, and thus the marginal utility is equal for every extra euro earned. On the contrary, consumers have a concave utility function in general, which means that they are assumed to be risk-averse. A concave utility function induces that a gain in wealth conveys a smaller increase to utility than the reduction in utility imparted by a loss in wealth of equal magnitude, that is, *diminishing marginal utility.* Another property of a concave utility function is that a customer with a low valuation may value one unit of money more than a customer with a high valuation. In other words, the marginal utility of a euro to a 'poor' customer is likely to exceed the marginal utility of a euro to a 'rich' customer [17, Chapter VII]. Daniel Bernoulli [6] first proposed a utility function that is equal to the natural logarithm of wealth. A logarithmic function is monotonically increasing and the marginal utility function is monotonically decreasing, which are the two basic mathematical properties that consumer utility functions have to satisfy [4]. In the words of Savage [16], "no other function has been suggested as a better prototype for Everyman's utility function". Based on this discussion, we model the utility of customer $j \in J$ as $\ln(b_j - p(j) + 1)w_j$, where the addition of 1 is solely to have a positive function. Conclusively, the social welfare, and thus the government's objective function, is defined as $\sum_{j \in J}(\ln(b_j - p(j) + 1) + p(j) - c(j))w_j$. In Section 3.1, we furthermore include a tax payment for violating the price regulative constraints, similar as in Section 2. However, this does not change the social welfare function, as the tax is paid by the producer to the government. Thus, the producer's utility is decreased by the same amount as the government's utility is increased, also known as *transferable utility.* This assumption is justified when the producer and the government have a common currency that is valued equally by both. Another reason not to include the tax payment in the social welfare is that it is a punishment to the producer, not to the society as a whole.

3.1 Price Regulation by Tax

In this section, the price regulative constraints $p_k \leq \alpha_k$ are not enforced by law, but their violation is penalized via tax (or penalty) τ_1. We define $\bar{g}_k(w) = \sum_{j \in J} d_{jk} w_j$ for all items $k \in K$. The penalty function is defined as

$$f_k^{\text{TAX}}(p, w, \alpha, \tau) = g_k(p, \alpha_k)^+ \cdot \tau_1 \cdot \bar{g}_k(w) = \sum_{j \in J} d_{jk}(p_k - \alpha_k)^+ \tau_1 w_j, \quad \forall\, k \in K.$$

Thus, given vector α, the government determines tax level τ_1 to maximize social welfare.

$$\begin{array}{lcl} \max_{\tau_1 \geq 0} & f^{\text{G}} & = \sum_{j \in J}(\ln(b_j - p(j) + 1) + p(j) - c(j))w_j \\ \max_{p \in \Re_+^m} & f^{\text{P}} - f^{\text{TAX}} & = \sum_{j \in J}\left(p(j) - c(j) - \sum_{k \in K} d_{jk}(p_k - \alpha_k)^+ \tau_1\right) w_j \\ \max_{w_j \in \{0,1\}} & f_j^{\text{C}} w_j & = (b_j - p(j))w_j \end{array}$$

For any given α_k, $k \in K$, consider the arrangement of linear equalities defined in $\Re^m$ by the *valuation constraints* $p(j) = b_j$ for every customer $j \in J$ (that is, $f_j^{\text{C}} = 0$), nonnegativity constraints $p_k = 0$ and price regulative constraints $p_k = \alpha_k$ for every item $k \in K$.

Definition 2. *A* vertex $v \in V$ *is defined as a price vector* $p = (p_1, \ldots, p_m)$ *that satisfies* m *linearly independent constraints out of the* $n + 2m$ *constraints* $p(j) = b_j$, $j \in J$, $p_k = \alpha_k$, *and* $p_k = 0$, $k = 1, \ldots, m$.

As a direct consequence of Theorem 1, for any given vector α and the given social welfare function f^{G}, profit maximizing price vectors $p = (p_1, \ldots, p_m)$ can only be vertices as defined in Definition 2. More specifically, one can easily check that the necessary Karush-Kuhn-Tucker conditions (see e.g. [5]) do not hold in any point except the vertices.

Notice that a vertex $v \in V$ is most preferable to the producer if the profit after tax at this vertex, $f^{\mathrm{P}}(p^{(v)}, w^{(v)}) - f^{\mathrm{TAX}}(p^{(v)}, w^{(v)}, \alpha, \tau)$, is at least as high as at any other vertex. Let $W^{(v)} = \{j \in J : w_j^{(v)} = 1\}$. More precisely, for every vertex $u \in V \setminus v$, the tax level τ_1 must be such that

$$\textstyle\sum_{j \in W^{(v)}} p^{(v)}(j) - c(j) - \sum\limits_{k \in K} d_{jk}(p_k^{(v)} - \alpha_k)^+ \tau_1 \geq$$
$$\textstyle\sum_{j \in W^{(u)}} p^{(u)}(j) - c(j) - \sum_{k \in K} d_{jk}(p_k^{(u)} - \alpha_k)^+ \tau_1.$$

Note that all terms except τ_1 in the above inequality are known, as α_k is given and $p_k^{(v)}$ is defined for all $k \in K$ and $v \in V$. Let us denote $T^{(v)} = \sum_{j \in W^{(v)}} \sum_{k \in K} d_{jk}(p_k^{(v)} - \alpha_k)^+$. We rewrite the latter inequality and solve the following feasibility linear program (LP2) below for each vertex $v \in V$.

$$\left(T^{(v)} - T^{(u)}\right) \tau_1 \leq \sum_{j \in W^{(v)}} \left(p^{(v)}(j) - c(j)\right) - \sum_{j \in W^{(u)}} \left(p^{(u)}(j) - c(j)\right) \; \forall\, u \in V \setminus v$$
$$\tau_1 \geq 0.$$

Let $V^* \subseteq V$ be the set of vertices for which the above linear program has a feasible solution. Then, among all vertices in V^* we select the vertex v^* with the highest social welfare $\sum_{j \in W^{(v^*)}} \ln(b_j - p^{(v^*)}(j) + 1) + p^{(v^*)}(j) - c(j)$. The tax level τ_1 is obtained as a solution to the linear program for this particular vertex v^*. So we have proved:

Theorem 4. *For given price regulative constraints $p_k \leq \alpha_k$, the tax level τ_1 that maximizes the total social welfare, and the corresponding profit maximizing prices can be computed in polynomial time, given that the number of distinct item types m is constant.*

3.2 Parameter Optimization

So far we assumed *given* values of the price regulative parameters $\alpha_1, \ldots, \alpha_m$. In this section we optimize these parameters under the regulation that the producer sets the price $p_k \leq \alpha_k$ for all $k \in K$. Hereto, we use the following model.

$$\begin{array}{ccr}
\max_{\alpha \in \Re_+^m} & \sum_{j \in J} \left(\ln(b_j - p(j) + 1) + p(j) - c(j)\right) w_j & \\
\max_{p \in \Re_+^m} & \sum_{j \in J} \left(p(j) - c(j) - \sum_{k \in K} d_{jk}(p_k - \alpha_k)^+ \tau_1\right) w_j & \\
\max_{w_j \in \{0,1\}} & (b_j - p(j)) w_j & \forall\, j \in J \\
s.t. & p_k \leq \alpha_k & \forall\, k \in K
\end{array}$$

Since parameters α_k, $k \in K$, are not given, let V denote the set of vertices as defined in Definition 1. For every vertex $v \in V$, let $\alpha_k = p_k^{(v)}$ for all $k \in K$. Let $U = \{u \in V : p_k \leq \alpha_k, \forall\, k \in K\}$. Then, vertex $u \in U$ is most preferable to the producer if the profit $\sum_{j \in W^{(u)}} p^{(u)}(j) - c(j) \geq \sum_{j \in W^{(u')}} p^{(u')}(j) - c(j)$ for all $u' \in U \setminus u$. Among all vertices that are most preferable given set U, we select the one with the highest social welfare and set the α-parameters accordingly.

Theorem 5. *For the regulation that forbids the producer to violate the constraints, the parameter vector α that maximizes the total social welfare, and the corresponding profit maximizing prices can be computed in polynomial time, given that the number of distinct item types m is constant.*

4 Computational results

In the summer of 2007, the European Commission decided to implement a EU-wide ceiling on the international roaming tariffs. The maximum price for calling from abroad is € 0.5831, for receiving a call abroad is € 0.2856, and the maximum price an operator may charge another operator for using the network is € 0.3570. This latter is the cost of the operator for providing roaming service to the customers. In summer 2008, these prices are lowered to € 0.5474, € 0.2618 and € 0.3332, repectively. And in summer 2009, they will decrease even further to € 0.5117, € 0.2261 and € 0.3094. The goal of this practical study is to analyze the effect of properly chosen parameters on social welfare, and the advantage of using taxes instead of forbidding to violate the price regulation. We use data from a telephone operator containing the phone usage of customers with a prepaid subscription during one month, March 2007, thus this data set comes from the period before the introduction of the price regulation. The data contains, for each customer, the number of minutes and times each customer uses the mobile phone for different actions (e.g. calling within the home country or abroad, sending a text message, etc.). We determine for every customer which operator in the telephone market offers the cheapest possible total price for her complete contract. This price determines her *valuation* b_j. For this study, even though the data contains more information, we focus on optimizing the prices for roaming only, namely calling and receiving a call abroad. This is to say, we consider a problem in dimension 2, with prices p_1 for calling abroad, and p_2 for receiving a call abroad. We also impose the constraint that the price for receiving a call should not exceed the price for placing a call ($p_1 \geq p_2$). If customer $j \in J$ requests to call d_{j1} minutes from abroad and receives calls abroad for d_{j2} minutes, within a month, the price customer j has to pay is $p(j) = d_{1j}p_1 + d_{2j}p_2$.

4.1 Experiments

We apply the model and techniques from Section 3 to a data set containing 1366 customers. Also, we create one random sample out of this data set containing 500 customers. In the application of the first algorithm, we use the current price regulations imposed by the European Commission as described in the first paragraph of this section. The costs for calling from abroad (c_1) is also retrieved from the table, and the cost for the operator for a customer to receive a call, is half of this. We forbid the operator to violate this price regulation by law (Law); i.e., a penalty $\tau_1 = \infty$. Second, we keep the price regulative parameters α_k as they are, but now we find a tax level $\tau_1 \geq 0$ which maximizes social welfare (Tax). Note that there can be a range of feasible tax levels achieving the maximal social welfare. This effect was also observed in our results. Third, we compute the optimal social welfare by optimizing over the price regulative parameters α_k (OPT).

	Year	LAW	TAX	OPT
Complete data set	2007	2834.20	3315.77	3315.77
$n = 1366$	2008	2797.51	3443.59	3443.59
	2009	2809.15	3571.40	3571.40
Sample	2007	878.69	1030.63	1038.66
$n = 500$	2008	887.15	1054.92	1059.21
	2009	895.02	1074.73	1079.76

Table 1. Social welfare obtained using different algorithms with price regulations on making and receiving calls abroad.

Table 1 shows the total social welfare for all instances. A complete overview of the results is deferred to the full version of this paper. We summarize our conclusions from these results as follows. Introducing a tax (TAX) instead of enforcing the price regulation by law (LAW) leads to an increase in the social welfare. This suggests that a more liberal price regulative policy might have the potential to improve social welfare. The tax levels that are obtained using the algorithm are non trivial. In extreme cases (not observed here, though) a producer might be able to participate in the market where it would not be profitable to do so if violation of price regulations was forbidden. Not surprisingly, the social welfare is maximal when the α-parameters are optimized (OPT). In the sample, the social welfare is strictly larger in the latter case. For the complete data set, both algorithms yield the same social welfare. Concluding, it seems that the current EU practice does not yield the optimum.

5 Conclusion

First, we think it is an interesting result in its own that the given three-level program can indeed be solved in polynomial time by making use of linear programming techniques. Even though techniques are comparably simple and crucially use the fact that the dimension m is constant, we believe it is not straightforward to come up with a polynomial time algorithm.

Second, on the more economic side, our computational results suggest that a more liberal price regulative policy, namely taxation instead of regulation by law may lead to an increase in social welfare. But of course, this conclusion cannot be made hard as it depends very much on the choice of the social welfare function.

References

1. *European commission, information society, media directorate general: EU roaming regulation: Using your mobile abroad in the EU is now much cheaper (factsheet 59)*, June 2007.
2. G. Aggarwal, T. Feder, R. Motwani, and A. Zhu, *Algorithms for multi-product pricing*, Automata, Languages and Programming - ICALP 2004 (J. Díaz, J. Karhumäki, A. Lepistö, and D. Sannella, eds.), Lecture Notes in Computer Science, vol. 3142, Springer, 2004, pp. 72–83.
3. M. F. Balcan and A. Blum, *Approximation algorithms and online mechanisms for item pricing*, Proceedings of the 7th ACM Conference on Electronic Commerce, ACM, 2006, pp. 29–35.
4. J. Baz and G. Chacko, *Financial derivatives: pricing, applications, and mathematics*, Cambridge University Press, 2004.
5. M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear programming: Theory and algorithms*, 2 ed., John Wiley & Sons, Inc., 1993.
6. D. Bernoulli, *Specimen theoriae novae de mensura sortis (Translated from Latin into English by L. Sommer in 1954)*, Commentarii Academiae Scientiarium Imperialis Petropolitanae, vol. V, 1738, pp. 175–192.
7. P. Briest and P. Krysta, *Single-minded unlimited supply pricing on sparse instances*, Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms, ACM-SIAM, 2006, pp. 1093–1102.
8. E. D. Demaine, U. Feige, M.T. Hajiaghayi, and M. R. Salavatipour, *Combination can be hard: Approximability of the unique coverage problem*, Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms, ACM-SIAM, 2006, pp. 162–171.
9. K. Elbassioni, R. Sitters, and Y. Zhang, *A quasi-ptas for profit-maximizing pricing on line graphs*, Proceedings of the 15th Annual European Symposium on Algorithms (L. Arge and E. Welzl, eds.), Lecture Notes in Computer Science, vol. 4698, Springer, 2007, pp. 451–462.
10. A. Grigoriev, J. van Loon, R. Sitters, and M. Uetz, *Optimal pricing of capacitated networks*, Networks (2008, to appear).
11. A. Grigoriev, J. van Loon, M. Sviridenko, M. Uetz, and T. Vredeveld, *Bundle pricing with comparable items*, Algorithms - ESA 2007 (L. Arge, M. Hoffmann, and E. Welzl, eds.), Lecture Notes in Computer Science, vol. 4698, Springer, 2007, pp. 475–486.
12. ———, *Optimal bundle pricing with monotonicity constraint*, Operations Research Letters **36** (2008), 609–614.
13. V. Guruswami, J. D. Hartline, A. R. Karlin, D. Kempe, C. Kenyon, and F. McSherry, *On profit-maximizing envy-free pricing*, Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms, ACM-SIAM, 2005, pp. 1164–1173.
14. J. D. Hartline and V. Koltun, *Near-optimal pricing in near-linear time*, Algorithms and Data Structures - WADS 2005 (F. K. H. A. Dehne, A. López-Ortiz, and J.-R. Sack, eds.), Lecture Notes in Computer Sciences, vol. 3608, Springer, 2005, pp. 422–431.
15. P. Marcotte and G. Savard, *Graph theory and combinatorial optimization*, ch. 7 - Bilevel Programming: A Combinatorial Perspective, pp. 191–218, Springer, 2005.
16. L. J. Savage, *The foundations of statistics*, 2 ed., Courier Dover Publications, 1972.
17. S. Shavel, *Foundations of economic analysis of law*, Harvard University Press, 2004.