Mathematical Programming Approach to Multidimensional Mechanism Design for Single Machine Scheduling

(Extended Abstract)

Jelle Duives* Marc Uetz*

We consider the following optimal mechanism design problem in single machine scheduling. Given are n jobs which we regard as selfish agents. Jobs need to be processed non-preemptively on a single machine. Each job j has processing time p_j and a disutility w_j for waiting one unit of time. This defines the 2-dimensional type $t_j = (w_j, p_j)$ of a job j.

The allocation rule f is a mapping of type profiles $t = (t_j)_{j=1,\dots,n}$ to the set of all $n!$ schedules. If in a given schedule σ, $S_j(\sigma)$ denotes the starting time of job j, the jobs valuation for that schedule is $-w_j S_j(\sigma)$. We assume that the mechanism designer needs to compensate jobs for waiting in the form of a payment π_j, such that $\pi_j - w_j S_j \geq 0$. We consider this problem in a Bayes-Nash setting, that is, given are publicly known, discrete probability distributions describing the jobs' possible types. Our goal is to find a (Bayes-Nash) incentive compatible mechanism that is optimal, which is a mechanism that minimizes the total expected payment $\sum_j E\pi_j$.

This Problem has been considered earlier in a paper by Heydenreich et al. [3], where mainly the special case of 1-dimensional type spaces has been analyzed (that is, public processing times p_j and private w_j). Also in that paper, an example has been proposed to show that optimal mechanisms in the 2-dimensional case in general do not satisfy a condition called IIA, independence of irrelevant alternatives [1]. However, that example was flawed.

Mixed Integer Programming Approach and Results

Motivated by the questions left open in [3], we were interested in getting more insight into properties of (optimal) mechanisms for the 2-dimensional case. In particular, in search for succinct characterizations of optimal mechanisms, at least for special cases, IIA is about the minimal property that such a mechanism should have. In other words, if (optimal) mechanisms are not IIA, this gives a hint towards intractability of the optimal mechanism design problem. Hence our particular interest in the IIA condition.

Moreover, it can been shown that for the 1-dimensional problem, Bayes-Nash incentive compatibility (BIC) and Dominant Strategy incentive compatibility (DIC) is equivalent in the sense that any mechanism that is BIC implementable, is also DIC implementable with the same expected total payment [**?**]. Such a result is also known for the single-object auction setting; see Manelli and Vincent [7]. However, Gerskhov et al. [6] show that such a result can only be valid in restrictive environments, as in general, BIC and DIC implementability are not equivalent as soon as there are at least three possible outcomes. Therefore, we are also interested in analysing

*j.duives@student.utwente.nl, m.uetz@utwente.nl. Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands

[1] An allocation rule f satisfies independence of irrelevant alternatives if the relative order of any two jobs j_1 and j_2 is the same in the schedules $f(t^1)$ and $f(t^2)$ for any two type profiles t^1, t^2 that differ only in the types of agents other than j_1 and j_2.

both BIC and DIC implementability in the 2-dimensional mechanism design problem for the specific scheduling problem considered here.

This is the context of our work, and our contribution is as follows. In the flavour of recent work on automated mechanism design as proposed by Conitzer and Sandholm, see [1, 5], we formulate the optimal mechanism design problem for this scheduling application as Mixed Integer Linear Programming Problem (MIP). Here, the payment scheme gives rise to continuous variables $\pi_j \geq 0$, and the allocation rule f gives rise to 0-1-variables as follows.

$$f_{t,\sigma} = \begin{cases} 1 & \text{type profile } t \text{ is assigned schedule } \sigma \\ 0 & \text{otherwise} \end{cases}$$

Using these variables, individual rationality and (Bayes-Nash) incentive compatibility can easily be written as linear constraints. What is particularly charming about this formulation is that we can express the IIA condition as a quadratic constraint, which can be linearised using standard methods. Moreover, the formulation is universal enough to be used for both Bayes-Nash and Dominant Strategy incentive compatibility. Concerning the optimal mechanism for the mentioned 2-dimensional scheduling application we have found the following.

Theorem 1 *For the 2-dimensional optimal mechanism design problem as sketched above*

(i) the optimal mechanism for the Bayes-Nash setting does in general not satisfy the IIA condition

(ii) Bayes-Nash incentive compatibility and Dominant Strategy incentive compatibility are not equivalent

Both results are in fact obtained with minimal instances consisting of only 3 jobs. This is minimal as for instances with 2 jobs, (*i*) all mechanisms are trivially IIA, and (*ii*) Bayes-Nash incentive compatibility and Dominant Strategy incentive compatibility are equivalent, as there are only 2 possible outcomes [6]. Also note that, regarding (*i*), for the 1-dimensional problem the optimal mechanism is a modification of Smith's rule which clearly does satisfy the IIA condition. Finally, regarding (*ii*), the 1-dimensional problem does have the property that BIC and DIC are equivalent [4]. The counterexamples for the 2-dimensional case have been found by generating instances systematically at random, and computing optimal mechanisms (with or without enforcing IIA, Bayes-Nash and Dominant Strategy incentive compatible) for each of them.

Current and Future Work

Although our MIP formulation is too large to be used for solving real-world instances, we are able to solve problems with up to 4 jobs in a matter of seconds. Hence, the MIP approach serves well in testing and generating hypotheses. One such hypothesis, which is currently backed by empirical evidence but lacks a formal proof is the following.

Conjecture 2 *For the 2-dimensional optimal mechanism design problem as sketched above where the types of jobs are generated by a product distribution rather than an arbitrary distribution, the optimal mechanism for the Bayes-Nash setting satisfies the IIA condition.*

Apart from proving this conjecture, further ambitions for future work are analyzing the relative degradation in costs comparing Bayes-Nash and Dominant Strategy implementations in terms of a worst case analysis, doing the same for IIA implementations, as well as proving the

2-dimensional optimal mechanism design problem for this specific scheduling problem as computationally hard (a result that does not follow from [1]). Finally, enhancing the MIP approach in order to make it computationally feasible is an obvious next step as well; see also [2]. To that end, we are also experimenting with alternative MIP formulations.

References

[1] V. Conitzer and T. Sandholm. Complexity of mechanism design. In: Proc. 18th Annual Conference on Uncertainty in Artificial Intelligence (UAI 2002), 103-110.

[2] M. Guo and V. Conitzer. Computationally Feasible Automated Mechanism Design: General Approach and Case Studies. In: Proc. National Conference on Artificial Intelligence (AAAI 2010), 1676-1679.

[3] B. Heydenreich, D. Mishra, R. Müller, and M. Uetz. Optimal Mechanisms for Single Machine Scheduling. In: Internet and Network Economics (WINE 2008), C. Papadimitriou and S. Zhang (eds.), Lecture Notes in Computer Science 5385, 2008, 414-425.

[4] B. Heydenreich, D. Mishra, R. Müller, and M. Uetz. Optimal Mechanisms for Scheduling. Manuscript, submitted 2010.

[5] T. Sandholm. Automated Mechanism Desgin: A New Application Area for Search Algorithms. Invited paper in: Proc. International Conference on Principles and Practice of Constraint Programming (CP 2003), Lecture Notes in Computer Science 2833, 2003, 19-36.

[6] A. Gershkov, B. Moldovanu, and X. Shi. Bayesian and Dominant Strategy Implementation Revisited. Manuscript, 2011. Retrieved from http://pluto.huji.ac.il/~alexg/.

[7] A.M. Manelli and D.R. Vincent. Bayesian and Dominant Strategy Implementation in the Independent Private Values Model. *Econometrica*, 78(6):1905-1938, 2010.